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## A Non-Gaussian Approach to Risk Measures

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### Abstract

Reliable calculations of financial risk require that the fat-tailed nature of prices changes is included in risk measures. To this end, a non-Gaussian approach to financial risk management is presented, modeling the power-law tails of the returns distribution in terms of a Student- $t$  distribution. Non-Gaussian closed-form solutions for Value-at-Risk and Expected Shortfall are obtained and standard formulae known in the literature under the normality assumption are recovered as a special case. The implications of the approach for risk management are demonstrated through an empirical analysis of financial time series from the Italian stock market and in comparison with the results of the most widely used procedures of quantitative finance. Particular attention is paid to quantify the size of the errors affecting the market risk measures obtained according to different methodologies, by employing a bootstrap technique.

*Key words:* Econophysics; Financial risk; Risk measures; Fat-tailed distributions; Bootstrap

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## 1 Introduction

A topic of increasing importance in modern economy and society is the development of reliable methods of measuring and controlling financial risks. According to the new capital adequacy framework, commonly known as Basel II accord [1], any financial institution has to meet stringent capital requirements in order to cover the various sources of risk that they incur as a result of their normal operation. Basically, three different categories of risk are of interest: credit risk, operational risk and market risk. In particular, market risk concerns the hazard of losing money due to the fluctuations of the prices of those instruments entering a financial portfolio and is, therefore, particularly important for financial risk management.

In the financial industry today, the most widely used measure to manage market risk is Value-at-Risk (VaR) [2,3]. In short, VaR refers to the maximum potential loss over a given period at a certain confidence level and can be used to measure the risk of individual assets and portfolios of assets as well. Because of its conceptual simplicity, VaR has become a standard component in the methodology of academics and financial practitioners. Moreover, VaR generally provides a reasonably accurate estimate of risk at a reasonable computational time. Still, as discussed in the literature [3,4], VaR suffers from some inconsistencies: first, it can violate the sub-additivity rule for portfolio risk, which is a required property for any consistent measure of risk, and, secondly, it doesn't quantify the typical loss incurred when the risk threshold is exceeded. To overcome the drawbacks of VaR, the Expected Shortfall (or Conditional VaR) is introduced, and sometimes used in financial risk management, as a more coherent measure of risk.

Three main approaches are known in the literature and used in practice for calculating VaR and Expected Shortfall. The first method consists in assuming some probability distribution function for price changes and calculating the risk measures as closed-form solutions. This approach is called parametric or analytical and is easy to implement since analytical expressions can often be obtained. The parametric approach usually relies on the (log)normality assumption for the returns distribution, although some analytical results using non-Gaussian functional forms are available in the literature [5,6]. Actually, it is well known that empirical price returns, especially in the limit of high frequency, do not follow the Gaussian paradigm and are characterized by heavier tails and a higher peak than a normal distribution. In order to capture the leptokurtic (fat-tailed) nature of price returns, the historical simulation approach is often used as an alternative to the parametric method. It employs recent historical data and risk measures are derived from the percentiles of the distribution of real data. This method is potentially the most accurate because it accounts for the real statistics of price changes but it is computationally

quite demanding (especially when applied to large portfolios) and absolutely depending on the past history of empirical data. A third approach consists in Monte Carlo simulations of the stochastic dynamics of a given model for stock price returns and in calculating risk measures according to Monte Carlo statistics. This method, however, requires very intensive simulations to achieve risk measures predictions with acceptable numerical errors.

As a result of the present situation, reliable and possibly fast methods to calculate financial risk are strongly demanded. Inspired by this motivation, the aim of this paper is to present a non-Gaussian approach to market risk management and to describe its potentials, as well as limitations, in comparison with standard procedures used in financial analysis. To capture the excess of kurtosis of empirical data with respect to the normal distribution, the statistics of price changes is modeled in terms of a Student- $t$  distribution, which is known to approximate with good accuracy the distribution derived from market data at a given time horizon [3,7] and is widely used in the financial literature. In the econophysics literature, the Student- $t$  distribution is also known as Tsallis distribution, emerging within the framework of statistical physics [8]. It has been shown in various studies [8,9] that the distribution of returns can be modeled quite well by a Tsallis distribution, which, for this reason, has been already used in a number of financial applications, ranging from option pricing [10] to risk analysis [11]. However, with respect to the investigation of Ref. [11], we include in our analysis the study of the Expected Shortfall and we present, in the spirit of a parametric approach, analytical expressions for the risk measures in order to provide accessible results for a simple practical implementation. At a variance of the recent calculation in Ref. [6], where analytical results for risk measures using Student- $t$  distributions are presented, we critically investigate the implications of our non-Gaussian analytical solutions on the basis of an empirical analysis of financial data and we perform detailed comparisons with the results of widely used procedures.

The paper is organized as follows. In Section 2 non-Gaussian closed-form expressions for VaR and Expected Shortfall are derived as generalizations of the analytical formulae known in the literature under the normality assumption. It is also shown how the standard Gaussian formulae of the parametric approach are recovered, in the appropriate limit, as a special case. In Section 3 an empirical analysis of daily returns series from the Italian stock market is performed, in order to constrain the Student- $t$  parameters entering the formulae of Section 2 and to describe the ingredients needed for the forthcoming risk analysis. The latter is carried out in Section 4. The implications of the parametric non-Gaussian approach for VaR and Expected Shortfall are shown in Section 4 and compared with the results of the parametric normal method, of its improved version known as RiskMetrics methodology and of the historical simulation. Particular attention is paid to quantify the size of the errors affecting the various risk measures, by employing a bootstrap technique. Con-

clusions and possible perspectives are drawn in Section 5.

## 2 Non-Gaussian closed-form expressions for risk measures

Value-at-Risk is referred to the probability of extreme losses in a portfolio value due to adverse market movements. In particular, for a given significance level  $\mathcal{P}^*$  (typically 1% or 5%), VaR, usually denoted as  $\Lambda^*$ , is defined as the maximum potential loss over a fixed time horizon  $\Delta t$ . In terms of price changes  $\Delta S$ , or, equivalently, of returns  $R \doteq \Delta S/S$ , VaR can be computed as follows

$$\mathcal{P}^* \doteq \int_{-\infty}^{-\Lambda^*} d\Delta S \tilde{P}_{\Delta t}(\Delta S) = S \int_{-\infty}^{-\Lambda^*/S} dR P_{\Delta t}(R), \quad (1)$$

where  $\tilde{P}_{\Delta t}(\Delta S)$  and  $P_{\Delta t}(R)$  are the probability density functions (pdfs) for price changes and for returns over a time horizon  $\Delta t$ , respectively. For financial analysts, VaR has become the standard measure used to quantify market risk because it has the great advantage to aggregate several risk component into a single number. In spite of its conceptual simplicity, VaR shows two main drawbacks: it is not necessary subadditive and it does not quantify the size of the potential loss when the threshold  $\Lambda^*$  is exceeded.

A quantity that does not suffer of these disadvantages is the so called Expected Shortfall (ES) or Conditional VaR (CVaR),  $E^*$ . It is defined as

$$E^* \doteq \frac{1}{\mathcal{P}^*} \int_{-\infty}^{-\Lambda^*} d\Delta S (-\Delta S) \tilde{P}_{\Delta t}(\Delta S) = \frac{S}{\mathcal{P}^*} \int_{-\infty}^{-\Lambda^*/S} dR (-R) P_{\Delta t}(R), \quad (2)$$

with  $\mathcal{P}^*$  and  $\Lambda^*$  as in Eq. (1).

The standard approach in the financial literature [2,12] is to assume the returns as normally distributed, with mean  $m$  and variance  $\sigma^2$ , i.e.  $R \sim \mathcal{N}(m, \sigma^2)$ . In that case, VaR and ES analytical expressions reduce to the following closed-form formulae

$$\Lambda^* = -mS_0 + \sigma S_0 \sqrt{2} \operatorname{erfc}^{-1}(2\mathcal{P}^*) \quad (3)$$

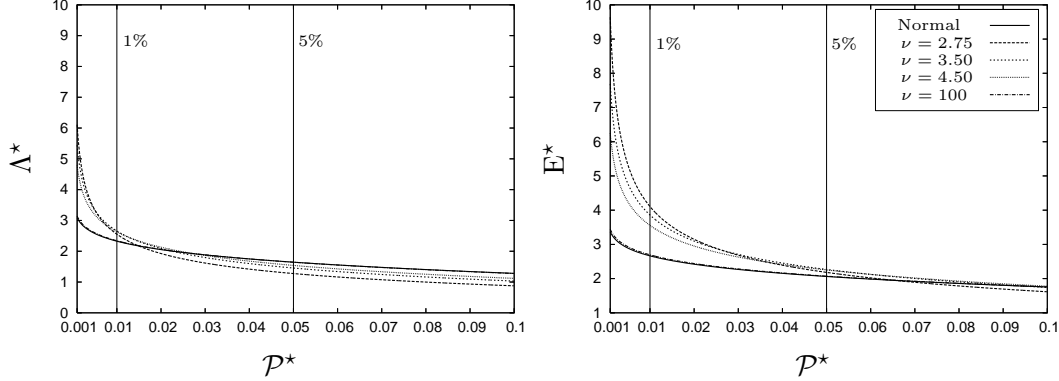
and

$$E^* = -mS_0 + \frac{\sigma S_0}{\mathcal{P}^*} \frac{1}{\sqrt{2\pi}} \exp\{-[\operatorname{erfc}^{-1}(2\mathcal{P}^*)]^2\}, \quad (4)$$

where  $S_0$  is the spot price and  $\operatorname{erfc}^{-1}$  is the inverse of the complementary error function [13].

However, it is well known from several studies, especially in the econophysics literature [3,7,14], that the normality hypothesis is often inadequate for daily returns and, more generally, for high-frequency stock price variations. A better

Fig. 1. Convergence of the VaR (left) and ES (right) Student- $t$  formulae toward Gaussian results when approaching the limit  $\nu \rightarrow +\infty$ .



agreement with data is obtained using leptokurtic distributions, such as truncated Lévy distributions or Student- $t$  ones. Despite this interesting feature, the former family has the main disadvantage that it is defined only through its characteristic function and we have no analytic expression for the pdfs [15]. Moreover, in order to compute the cumulative density function (cdf), which is a necessary ingredient of our analysis, we have to resort to numerical approximations. For the reasons above, to model the returns, we make use of a Student- $t$  distribution defined as

$$\mathcal{S}_{m,a}^\nu(R) = \frac{1}{B(\nu/2, 1/2)} \frac{a^\nu}{[a^2 + (R - m)^2]^{\frac{\nu+1}{2}}}, \quad (5)$$

where  $\nu \in (1, +\infty)$  is the tail index and  $B(\nu/2, 1/2)$  is the beta function. It is easy to verify that, for  $\nu > 2$ , the variance is given by  $\sigma^2 = a^2/(\nu - 2)$ , while, for  $\nu > 4$ , the excess kurtosis reduces to  $k = 6/(\nu - 4)$ . Under this assumption, we obtain closed-form generalized expression for VaR and ES given by

$$\Lambda^* = -mS_0 + \sigma S_0 \sqrt{\nu - 2} \sqrt{\frac{1 - \lambda^*}{\lambda^*}} \quad (6)$$

and

$$E^* = -mS_0 + \frac{\sigma S_0}{\mathcal{P}^* B(\nu/2, 1/2)} \frac{\sqrt{\nu - 2}}{\nu - 1} [\lambda^*]^{\frac{\nu-1}{2}}, \quad (7)$$

where  $\lambda^* \doteq I_{[\nu/2, 1/2]}^{-1}(2\mathcal{P}^*)$  and  $I_{[\nu/2, 1/2]}^{-1}$  is the inverse of the incomplete beta function, according to the definition of Ref. [13].

As shown in Fig. 1, we have checked numerically the convergence of formulae (6) and (7) to the Gaussian results (3) and (4), in the appropriate limit  $\nu \rightarrow +\infty$ . We chose  $\nu = 2.75, 3.5, 4.5, 100$  and  $m = 0$ ,  $\sigma S_0 = 1$ , but we checked that the value of these last parameters does not affect the convergence, as expected. As can be seen, the points corresponding to  $\nu = 100$  are almost coincident with the Gaussian predictions, demonstrating that our results correctly recover the Gaussian formulae as a special case. It is also worth

noting that each line, corresponding to a fixed  $\nu$ , crosses over the Gaussian one for a certain  $\mathcal{P}^*$ . Analogously, for a fixed  $\mathcal{P}^*$ , there exists a  $\nu_{\text{cross}}$  value whose line crosses the Gaussian result at that significance level. In the light of this observation, we report in Table 1 the values of  $\nu_{\text{cross}}$  corresponding to a given  $\mathcal{P}^*$  for both VaR and ES. As can be observed, the growth of  $\nu_{\text{cross}}$  with  $\mathcal{P}^*$  is very rapid for VaR, while for ES and for usually adopted significance values,  $\nu_{\text{cross}}$  keeps in the interval  $[2.09, 2.51]$ . From this point of view, VaR and ES are quite different measures of risk, since the crossover values for the latter are much more stable than those associated to the first one. This result can be interpreted as a consequence of ES as a more coherent risk measure than VaR.

Table 1

Values of  $\nu$  crossover for VaR and ES corresponding to different significance levels  $\mathcal{P}^*$ .

$\mathcal{P}^*$	1%	2%	3%	4%	5%
$\nu_{\text{cross}}(\text{VaR})$	2.44	3.21	5.28	32.38	$\gg 100$
$\nu_{\text{cross}}(\text{ES})$	2.09	2.18	2.28	2.38	2.51

### 3 Empirical analysis of financial data

The data sets used in our analysis consist of four financial time series, composed of  $N = 1000$  daily returns, from the Italian stock market. Two series are collections of data from the Italian assets Autostrade SpA and Telecom Italia (from May 15<sup>th</sup> 2001 to May 5<sup>th</sup> 2005), while the other two correspond to the financial indexes Mib30 and Mibtel (from March 27<sup>th</sup> 2002 to March 13<sup>th</sup> 2006). The data have been freely downloaded from Yahoo Finance [16].

Figure 2 shows a comparison between the historical complementary cumulative density function  $P_>$  of the negative daily returns and two theoretical fits obtained using Gaussian and Student- $t$  distributions. The parameters values of the fitted curves, as obtained according to the likelihood procedure described below, are displayed in Table 2. In principle, we could perform the fit according to different methods, but we have to balance between accuracy and computational time. Therefore, we estimate mean and variance as empirical moments, i.e.

$$m \doteq \frac{1}{N} \sum_{i=0}^{N-1} R_{t-i} \quad (8)$$

and

$$\sigma^2 \doteq \frac{1}{N-1} \sum_{i=0}^{N-1} (R_{t-i} - m)^2, \quad (9)$$

where  $\mathbf{R} \doteq (R_t, \dots, R_{t-N+1})$  is the  $N$ -dimensional vector of returns. Using the above  $m$  and  $\sigma$  values, we derive a standardized vector (with zero mean and unit variance)  $\mathbf{r} \doteq (r_t, \dots, r_{t-N+1})$ , where  $r_{t-i} \doteq (R_{t-i} - m)/\sigma$  for  $i = 0, \dots, N-1$ . In order to find the best value for the tail parameter  $\nu$ , we look for the argument that minimizes the negative log-likelihood, according to the formula

$$\nu = \operatorname{argmin}_{\nu > 2} \left[ - \sum_{i=0}^{N-1} \log \mathcal{S}_{0, \sqrt{\nu-2}}^{\nu}(r_{t-i}) \right], \quad (10)$$

where the constraint  $\nu > 2$  prevents the variance to be divergent and  $\mathcal{S}_{0, \sqrt{\nu-2}}^{\nu}$  is as in Eq. (5), with  $m = 0$  and  $a = \sqrt{\nu-2}$ . This apparently simple optimization problem can not be solved analytically. In fact, the normalization factor in the Eq. (5) does depend on the tail index  $\nu$  in a non trivial way. Actually, the beta function  $B(\nu/2, 1/2)$  only admits an integral representation and therefore we implemented a numerical algorithm to search for the minimum.

As shown in Section 2, the excess kurtosis  $k$  depends only on  $\nu$  and this provides an alternative and more efficient way to estimate the tail parameter [11]. However, this approach forces  $\nu$  to be bigger than 4, while from Table 2 it can be seen that all the exponents obtained in the likelihood-based approach are smaller than 3.5. For this reason, the implementation of the excess kurtosis method is inadequate for the time series under study here. In order to test the robustness of our results, we also performed a more general three-dimensional minimization procedure over the free parameters  $(m, \sigma, \nu)$ . The multidimensional optimization problem was solved by using the MINUIT program from CERN library [17]. The obtained numerical results are in full agreement with the previous ones, but the process is more computationally burden and more cumbersome, since it requires a lot care in avoiding troubles related to the appearing of local minima in the minimization strategy.

In Fig. 2 we show the cumulative distribution  $P_{>}$  obtained using the empirical parameters of Table 2. As expected, we measure daily volatilities of the order of 1% and quite negligible means ( $\sim 0.01\%$ ). The tail parameters fall in the range (2.9, 3.5), thus confirming the strong leptokurtic nature of the returns distributions, both for single assets and market indexes. The quality of our fit clearly emerges from Fig. 2, where one can see a very good agreement between Student- $t$  and historical complementary cdfs, while the Gaussian distribution fails to reproduce the data.

Before addressing a risk analysis in the next Section, it is worth mentioning, for completeness, that other approaches to model with accuracy the tail exponent of the returns cdfs are discussed in the literature. They are based on Extreme Value Theory [18] and Hill's estimator [19,20]. However, since they mainly focus on the tails, they require very long time series to accumulate sufficient statistics and are not considered in the present study.

Fig. 2. From top left clockwise: Autostrade SpA, Telecom Italia (from May 15<sup>th</sup> 2001 to May 5<sup>th</sup> 2005), Mibtel and Mib30 (from March 27<sup>th</sup> 2002 to March 13<sup>th</sup> 2005)  $P_{>}$  of negative daily returns. Points represent historical complementary cdf, while dashed and solid lines correspond to Gaussian and Student fits, respectively. The parameters values of the fitted curves are detailed in Table 2.

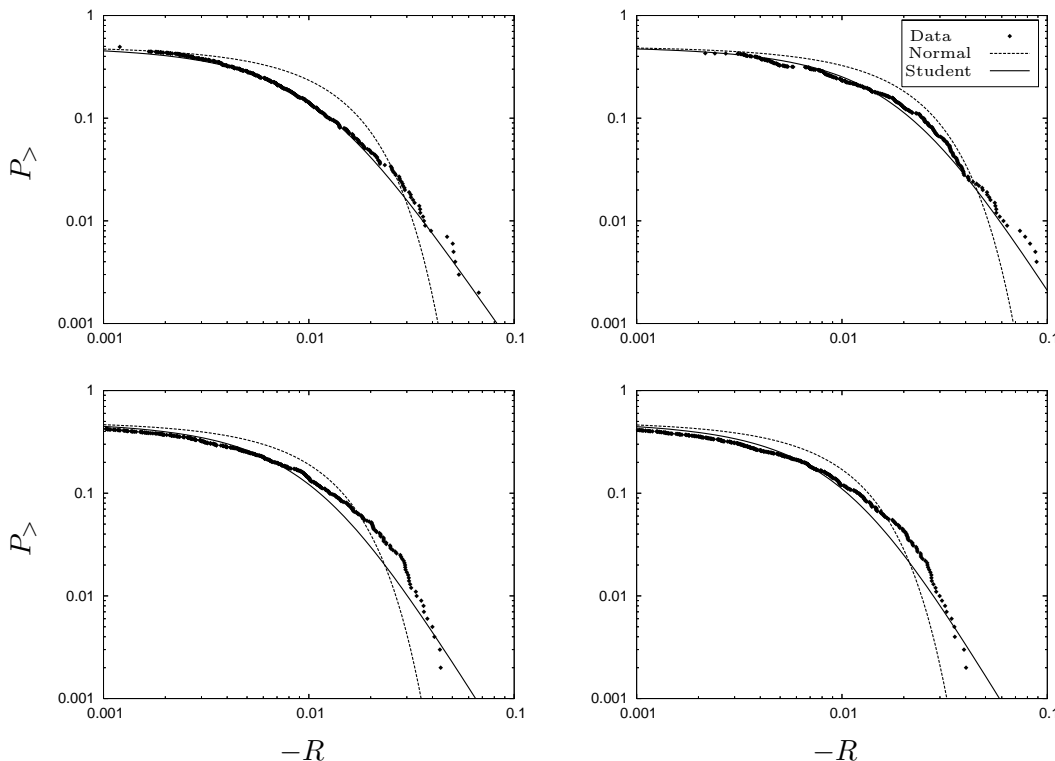


Table 2

Mean  $m$ , volatility  $\sigma$ , and tail exponent  $\nu$ , for Autostrade SpA, Telecom Italia, Mibtel and Mib30 time series.  $m$  and  $\sigma$  are estimated from empirical moments, while  $\nu$  is obtained through a negative log-likelihood minimization as in Eq. (10).

	$m$	$\sigma$	$\nu$
Autostrade	0.12%	1.38%	2.91
Telecom	-0.02%	2.23%	3.14
Mibtel	0.02%	1.03%	3.35
Mib30	0.02%	1.16%	3.22

## 4 Risk analysis

In this Section we present a comparison of the results obtained estimating the market risk through VaR and ES according to different methodologies. The standard approach is based on the normality assumption for the distribution of the returns. For this case we are provided of closed-form solutions, Eqs. (3) and (4), that depend on the two parameters  $m$  and  $\sigma$ . For the time series



under consideration, the effect of the mean, as shown before, is negligible, and the surviving parameter is the volatility  $\sigma$ . Several techniques are discussed in the literature to model and forecast volatility, based on stochastic volatility approaches [21], GARCH-like [22] and multifractal models [23]. They usually require very long time series (typically 300 high frequency returns per day over  $\sim 5 - 10$  years) and are quite demanding from a computational point of view. As discussed in Section 3, we limit our analysis to 1000 daily data and we estimate the volatility using the empirical second moment. In order to avoid the problem of a uniform weight for the returns, RiskMetrics introduces the use of an exponential weighted moving average of squared returns according to the formula [12]

$$\sigma_{t+1|t}^2 \doteq \frac{1 - \lambda}{1 - \lambda^{N+1}} \sum_{i=0}^{N-1} \lambda^i (R_{t-i} - m)^2, \quad (11)$$

where  $\lambda \in (0, 1]$  is a decay factor. The choice of  $\lambda$  depends on the time horizon and, for  $\Delta t = 1$  day,  $\lambda = 0.94$  is the usually adopted value [12].  $\sigma_{t+1|t}$  represents volatility estimate at time  $t$  conditional on the realized  $\mathbf{R}$ . If one considers Eq. (11) as the defining equation for an autoregressive process followed by  $\sigma_{t+1|t}$  (coupled with  $R_t = \sigma_t \epsilon_t$  with  $\epsilon_t \sim \text{i.i.d.}(0, 1)$ ), Refs. [24,25] provide reasons for the claimed good success of the RiskMetrics methodology. In order to relax standard assumption about the return pdf without loosing the advantages coming from a closed-form expression, we presented in Section 2 generalized formulae for VaR and ES based on a Student- $t$  modeling of price returns. In this framework, the tail index  $\nu$  emerges as a third relevant parameter, which is possible to constrain using a maximum likelihood technique, as previously described. As a benchmark of all our results, we also quote VaR and ES estimates following a historical approach, which is a procedure widely used in the practice. According to this approach, after ordering the  $N$  data in increasing order, we consider the  $[N\mathcal{P}^*]^{\text{th}}$  return  $R_{([N\mathcal{P}^*])}$  as an estimate for VaR and the empirical mean over first  $[N\mathcal{P}^*]$  returns as an estimate for ES <sup>1</sup>.

At a variance with respect to previous investigations [11,24], we also provide 68% confidence level (CL) intervals associated to the parameters. In this way we can estimate VaR and ES dispersion. To this extent, we implement a bootstrap technique [26]. Given the  $N$  measured returns, we generate  $M = 1000$  synthetic copies of  $\mathbf{R}$ ,  $\{\mathbf{R}_j^*\}$ , with  $j = 1, \dots, M$ , by random sampling with replacement according to the probability  $p = (1/N, \dots, 1/N)$ . For each  $\mathbf{R}_j^*$  we estimate the quantities of interest and we obtain bootstrap central values and

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<sup>1</sup> The symbol  $[ ]$  stands for integer part, while  $R_{(j)}$  is the standard notation for the  $j^{\text{th}}$  term of the order statistic of  $\mathbf{R}$ . Since  $N \gg 1$  we neglect the fact that the  $p^{\text{th}}$  entry is a biased estimator of the  $p/N$ -quantile, i.e.  $\mathbb{E}[R_{(p)}] = p/(N + 1)$ .

Table 3

Parameters values and bootstrap estimates for the 68% CL intervals for the time series as in Table 2.

	$m$	$\sigma$	$\sigma_{t+1 t}$	$\nu$	$R_{(10)}$
Autostrade	$0.12^{+0.04}_{-0.05}\%$	$1.38^{+0.08}_{-0.10}\%$	$1.83^{+0.31}_{-0.33}\%$	$2.91^{+0.20}_{-0.21}$	$-3.51^{+0.31}_{-0.15}\%$
Telecom	$-0.02^{+0.06}_{-0.07}\%$	$2.23^{+0.11}_{-0.11}\%$	$1.54^{+0.42}_{-0.47}\%$	$3.14^{+0.21}_{-0.22}$	$-6.14^{+0.87}_{-1.35}\%$
Mibtel	$0.02^{+0.02}_{-0.04}\%$	$1.03^{+0.03}_{-0.04}\%$	$0.69^{+0.19}_{-0.20}\%$	$3.35^{+0.18}_{-0.19}$	$-2.96^{+0.25}_{-0.24}\%$
Mib30	$0.02^{+0.03}_{-0.04}\%$	$1.16^{+0.03}_{-0.05}\%$	$0.72^{+0.22}_{-0.22}\%$	$3.22^{+0.15}_{-0.16}$	$-3.33^{+0.30}_{-0.25}\%$

confidence levels. For example, we use for the mean the relations

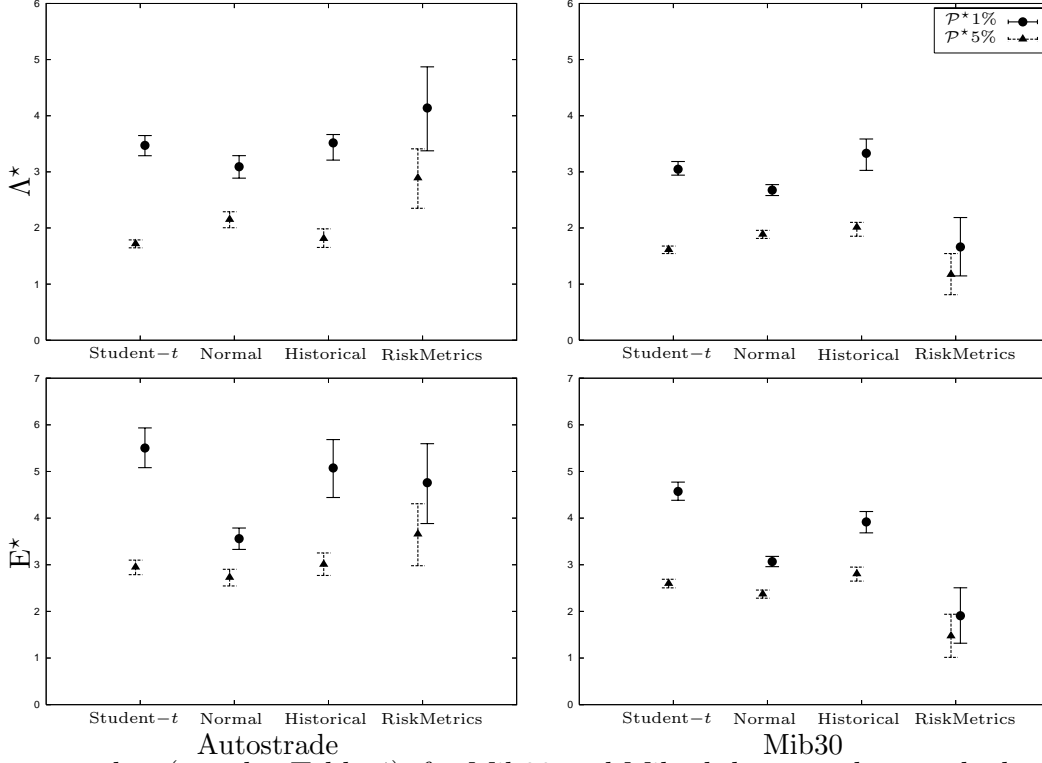
$$m_b^* \doteq \frac{1}{M} \sum_{j=1}^M m_j^* \quad \text{with} \quad m_j^* = \frac{1}{N} \sum_{i=0}^{N-1} (R_j^*)_{t-i} \quad (12)$$

and we define the  $1-2\alpha$  CL interval as  $[m_\alpha^*, m_{1-\alpha}^*]$ , with  $m_a^*$  such that  $P(m^* \leq m_a^*) = a$  and  $a = \alpha, 1 - \alpha$ . For 68% CL,  $\alpha = 16\%$ . In Fig. 3 and Tables 3, 4, 5 we quote results according to  $m - (m_b^* - m_\alpha^*) + (m_{1-\alpha}^* - m_b^*)$ . In this way, we use the bootstrap approach in order to estimate the dispersion of the mean around the measured value,  $m$ . Analogously for all the other parameters.

Table 3 shows central values and estimated 68% CL intervals for the daily returns series under study. These numerical results come from a straightforward application of the resampling technique. It is worth mentioning that it is possible, and sometimes necessary, to use improved versions of the bootstrap. As a rule of thumb, we consider the bootstrap approach accurate when, given a generic parameter, the difference between its empirical value and the bootstrap central value estimate is close to zero and 68% CL interval is symmetric to a good approximation. In our numerical simulation, we measured a systematic non zero bias for  $\sigma_{t+1|t}$  and from Table 3 it is quite evident the asymmetry of  $R_{([NP^*])}$  intervals for both Autostrade and Telecom data. We can, therefore, consider the corresponding CL intervals as a first approximation of the right ones, since bias and skewness corrections would require sophisticated and ad-hoc techniques [26], which are beyond the scope of the present work.

In Fig. 3 we show VaR and ES central values and 68% CL bars for Autostrade SpA and Mib30, corresponding to 1% and 5% significance level and according to the four methodologies previously described. In Tables 4 and 5 we detail all the numerical results, including also Telecom Italia and Mibtel data. As already noted in Ref. [24], at 5% significance level Student- $t$  and Normal approaches are substantially equivalent, but here such a statement sounds more statistically robust, thanks to the bootstrap 68% confidence levels and to the comparison with the historical simulation. At this significance level, we register for VaR a different behaviour between single assets and indexes. While assets show the best agreement between the Student- $t$  and historical

Fig. 3. VaR  $\Lambda^*$  (upper panel) and ES  $E^*$  (lower panel) central values with 68% CL intervals for Autostrade SpA (left) and for Mib30 (right), according to the four different methodologies discussed in the text. The significance level  $\mathcal{P}^*$  is fixed to 1% (circles, solid lines) and 5% (triangles, dashed lines).



approaches (see also Table 4), for Mib30 and Mibtel data we observe the best agreement between the Normal and historical methodology. In order to enforce this empirical evidence, it would be necessary to analyze additional time series to see to what extent this difference between assets and indexes holds. From Fig. 3, Table 4 and Table 5 it can also be seen that  $\Lambda^*$  and  $E^*$  central values calculated according to RiskMetrics methodology are quite fluctuating and characterized by the largest CL bars. The decreasing of  $\mathcal{P}^*$  traduces in a major differentiation of the different approaches. In general, we obtain the best agreement between the Student- $t$  approach and the historical simulation, both for  $\Lambda^*$  and  $E^*$ , whereas, as before, the RiskMetrics methodology overestimates or underestimates the results of the historical evaluation and is affected by rather large uncertainties.

To conclude, we would like to note that we expect, from the results shown in Fig. 1 and Table 1, that, for a fixed significance level, there exists a crossover value,  $\nu_{cross}$ , below which the generalized Student- $t$  VaR and ES formulae underestimate the Gaussian predictions. This effect was already mentioned in Ref. [11], but the analytical formulae here derived allow us to better characterize it. Under the hypothesis of a Student- $t$  distribution, the crossover value does not depend on the first and second moments and, therefore, the knowledge, for a given time series, of the tail exponent only is sufficient to conclude,

Table 4

Estimated VaR values (mean and 68% CL interval) for 1% and 5% significance levels from Autostrade SpA, Telecom Italia, Mib30 and Mibtel. For each time series, the results of Student- $t$  and Normal fit, historical simulation and RiskMetrics methodology are shown.

		Student- $t$	Normal	Historical	RiskMetrics
Autostrade	VaR 1%	$3.472^{+0.175}_{-0.185}$	$3.091^{+0.197}_{-0.204}$	$3.516^{+0.149}_{-0.306}$	$4.138^{+0.733}_{-0.764}$
	VaR 5%	$1.717^{+0.071}_{-0.071}$	$2.150^{+0.139}_{-0.145}$	$1.810^{+0.175}_{-0.156}$	$2.890^{+0.520}_{-0.540}$
Telecom	VaR 1%	$5.900^{+0.279}_{-0.230}$	$5.200^{+0.275}_{-0.277}$	$6.137^{+1.348}_{-0.866}$	$3.595^{+0.990}_{-1.085}$
	VaR 5%	$3.121^{+0.135}_{-0.137}$	$3.682^{+0.214}_{-0.202}$	$3.398^{+0.110}_{-0.127}$	$2.548^{+0.694}_{-0.777}$
Mib30	VaR 1%	$3.047^{+0.106}_{-0.105}$	$2.675^{+0.097}_{-0.096}$	$3.331^{+0.255}_{-0.304}$	$1.662^{+0.524}_{-0.516}$
	VaR 5%	$1.612^{+0.066}_{-0.067}$	$1.885^{+0.073}_{-0.072}$	$2.010^{+0.090}_{-0.157}$	$1.169^{+0.375}_{-0.358}$
Mibtel	VaR 1%	$2.718^{+0.097}_{-0.092}$	$2.378^{+0.088}_{-0.084}$	$2.967^{+0.240}_{-0.255}$	$1.581^{+0.453}_{-0.449}$
	VaR 5%	$1.454^{+0.062}_{-0.062}$	$1.674^{+0.065}_{-0.065}$	$1.811^{+0.150}_{-0.173}$	$1.110^{+0.324}_{-0.316}$

Table 5

Estimated ES values (mean and 68% CL interval) for 1% and 5% significance levels. Time series and methodologies as in Table 4.

		Student- $t$	Normal	Historical	RiskMetrics
Autostrade	ES 1%	$5.503^{+0.431}_{-0.421}$	$3.559^{+0.229}_{-0.231}$	$5.076^{+0.607}_{-0.634}$	$4.759^{+0.837}_{-0.876}$
	ES 5%	$2.946^{+0.153}_{-0.159}$	$2.727^{+0.175}_{-0.182}$	$3.006^{+0.248}_{-0.235}$	$3.655^{+0.653}_{-0.677}$
Telecom	ES 1%	$8.912^{+0.579}_{-0.583}$	$5.954^{+0.311}_{-0.310}$	$9.685^{+1.456}_{-1.475}$	$4.116^{+1.133}_{-1.250}$
	ES 5%	$5.035^{+0.242}_{-0.246}$	$4.613^{+0.248}_{-0.246}$	$5.320^{+0.478}_{-0.466}$	$3.190^{+0.879}_{-0.969}$
Mib30	ES 1%	$4.572^{+0.199}_{-0.191}$	$3.068^{+0.111}_{-0.109}$	$3.918^{+0.223}_{-0.234}$	$1.908^{+0.599}_{-0.590}$
	ES 5%	$2.596^{+0.093}_{-0.091}$	$2.369^{+0.088}_{-0.086}$	$2.804^{+0.145}_{-0.155}$	$1.471^{+0.467}_{-0.458}$
Mibtel	ES 1%	$4.021^{+0.179}_{-0.171}$	$2.728^{+0.099}_{-0.094}$	$3.501^{+0.215}_{-0.224}$	$1.815^{+0.524}_{-0.516}$
	ES 5%	$2.314^{+0.084}_{-0.081}$	$2.106^{+0.078}_{-0.077}$	$2.524^{+0.128}_{-0.136}$	$1.399^{+0.399}_{-0.400}$

a priori, whether the fat-tailed results for VaR and ES will underestimate or not the corresponding Gaussian estimates.

## 5 Conclusions

In this paper we have presented a careful analysis of financial market risk measures in terms of a non-Gaussian (Student) model for price fluctuations. We have derived closed-form parametric formulae for Value at Risk and Ex-

pected Shortfall that generalize standard expressions known in the literature under the normality assumption and can be used to obtain reliable estimates for the risk associated to a single asset or a portfolio of assets. The obtained non-Gaussian parametric formulae have been shown to be able to capture accurately the fat-tailed nature of financial data and, when specified in terms of the model parameters optimized by means of an empirical analysis of real daily returns series, have been found to be in good agreement with a full historical evaluation. Moreover, the risk measures obtained through our model show non negligible differences with respect to the widely used Normal and RiskMetrics methodologies, indicating that the approach may have helpful implications for practical applications in the field of financial risk management. We also proposed a bootstrap-based technique to estimate the size of the errors affecting the risk measures derived through the different procedures, in order to give a sound statistical meaning to our comparative analysis.

As far as possible perspectives are concerned, it would be interesting to investigate to what extent our conclusions, drawn from an analysis of a sample of Italian financial data, apply also to other financial markets. In particular, one could check whether, at a given significance level, statistically relevant differences are present between the results valid for a single asset and those relative to a portfolio of assets, as our analysis seems to indicate, at least for 5% VaR. Another interesting development concerns the comparison between the predictions for VaR and ES of our model with the corresponding ones derived by means of other statistical procedures to measure tail exponents known in the literature [18,19,20], as well as with the results from simulations of advanced models of the financial market dynamics, such as GARCH-like and multifractal models [22,23].

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